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COMMENT

Some remarks on the normalisation condition problem of the dressing method

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Abstract. In this small note we discuss the problem of the normalisation condition of the dressing method, which is crucial in finding exact solutions and in constructing Bäcklund transformation in the dressing method approach. We propose a method for finding a useful gauge-equivalent representation of the spectral problem which exhibits canonical normalisation. We illustrate this method in the Wadati–Konno–Ichikawa case.

The original approach to the so-called dressing method (DM) of soliton theory was formulated by Zakharov and Shabat (1979) and subsequently developed by Zakharov and Mikhalov (1978, 1980), Manakov *et al* (1980) etc.

In this approach the DM is some general solution proliferation technique. As far as pure N-soliton solutions (or their appropriate limits, i.e. rational solutions) are concerned, this technique reduces to the solution of algebraic equations and has been successfully applied to many soliton systems.

A second approach, called the Darboux matrix approach, has been originally invented to deduce from a given spectral problem (SP) a corresponding Bäcklund transformation (BT) (Benguria and Levi 1980, Bruschi and Ragnisco 1980, Levi and Ragnisco 1982, Levi *et al* 1982). The two approaches are, under quite general assumptions, equivalent.

In both approaches the essential problem is to find a proper normalisation condition for the Darboux matrix. The first, but not complete, general discussion has been given by Levi *et al* (1983).

Roughly speaking, the problem discussed looks as follows. Suppose we have a SP of the form

$$\psi_x = \boldsymbol{U}[q(x), r(x), \ldots ; q_x(x), r_x(x), \ldots ; \lambda]\psi \tag{1}$$

where $q, r, \ldots$ are potentials (or soliton fields at a fixed instant of time), λ is the spectral parameter assuming complex values, $\boldsymbol{U}$ and ψ are $n \times n$ matrix functions and a subscript always denotes differentiation. ψ is called a wavefunction of the SP (1). In the DM we assume the existence of another set of potentials $\tilde{q}, \tilde{r}, \ldots$ with the properties that the corresponding wavefunction

$$\tilde{\psi}_x = \boldsymbol{U}[\tilde{q}(x), \tilde{r}(x), \ldots ; \tilde{q}_x(x), \tilde{r}_x(x), \ldots ; \lambda]\tilde{\psi} \tag{2}$$

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can be related to ψ through the factorisation

$$\tilde{\psi}(x, \lambda) = \boldsymbol{D}(x, \lambda)\psi(x, \lambda) \tag{3}$$

where the $n \times n$ matrix function $\boldsymbol{D}(x, \lambda)$ is a polynomial in λ. When

$$\boldsymbol{D}(x, \lambda) = \lambda \boldsymbol{D}_1(x) + \boldsymbol{D}_0(x) \tag{4}$$

we call the matrix $\boldsymbol{D}$ 'an adding one-soliton Darboux matrix' (Levi *et al* 1982). ψ and $\tilde{\psi}$ are defined by the differential ordinary matrix equations (1) and (2); so we have a freedom in choosing them and consequently we have a freedom in choosing the $\boldsymbol{D}$ matrix. For instance we are free to replace $\boldsymbol{D}$ of (4) by $\boldsymbol{D}' = (1/\lambda)\boldsymbol{D} = \boldsymbol{D}_1 + \boldsymbol{D}_0/\lambda$; of course the limit of $\boldsymbol{D}'$ as $\lambda \to \infty$ *exists* and this limit is $\boldsymbol{D}_1(x)$.

In many cases this is true; it holds, for instance, for the Zakharov–Shabat SP (Zakharov and Shabat 1972, Ablowitz *et al* 1974). If this is true, we call the matrix $\boldsymbol{D}_1(x)$ a normalisation condition of the given SP. If $\boldsymbol{D}_1(x) = \boldsymbol{I}$, the identity matrix, we refer to this normalisation as a canonical one.

Now we are in a position to formulate a general *normalisation condition problem.* Let us go back to the general linear problem (1). To attack the normalisation condition problem: (i) firstly we should ask ourselves whether it is possible to construct a Darboux matrix $\boldsymbol{D}'$; (ii) secondly, when we can construct it and $\boldsymbol{D}' \neq \boldsymbol{I}$, whether we are able, by an appropriate gauge transformation, to reduce it to a simpler form (Levi *et al* 1983); (iii) finally, if it is not possible to find a non-trivial Darboux matrix $\boldsymbol{D}'$, we should try to invent another equivalent (for instance, gauge equivalent) representation of the SP (1) such that, in this new representation, we have a well defined normalisation condition $\boldsymbol{D}_1(x)$.

Having analysed previously (Levi *et al* 1983) the possibilities (i) and (ii), here we give a somewhat deeper analysis of the possibility (iii), proposing some general way for finding a more convenient representation of the SP (1); we shall illustrate the method in the case of the SP Wadati–Konno–Ichikawa (WKI). For the WKI SP this method will allow us to find a two-sided gauge transformation which reduces it to one with canonical normalisation.

The starting observation is the following. From (3) it follows that

$$\boldsymbol{D}(x, \lambda) = \tilde{\psi}(x, \lambda)\psi^{-1}(x, \lambda);$$

thus all the analysis and asymptotic properties of D (including existence or non-existence of the normalisation condition) are somehow coded in the same properties of the matrix wavefunction of the SP. Thus, in principle, we are able to decide whether it is possible or not to introduce a normalisation condition by means of the asymptotic analysis in λ of the matrix wavefunction ψ. To illustrate these concepts we start with the familiar example of the Zakharov–Shabat SP

$$\psi_x = \begin{pmatrix} \mathrm{i}\lambda & q(x) \\ r(x) & -\mathrm{i}\lambda \end{pmatrix}\psi. \tag{5}$$

In this case one can show that

$$\psi^{(+)}(x, \lambda; \tilde{q}, \tilde{r}) = \boldsymbol{D}'\psi^{(+)}(x, \lambda; q, r) \tag{6}$$

where $\psi^{(+)}(x, \lambda; q, r)$ (and similarly $\psi^{(+)}(x, \lambda; \tilde{q}, \tilde{r})$) is a particular solution of (5) with the property of being analytically extendable into the upper half-plane of the complex variable λ, such that

$$\psi^{(+)}(x, \lambda; q, r) \underset{\lambda \to \infty}{\sim} \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix} [I + O(\lambda^{-1})]. \tag{7}$$

From (6) and (7) it follows immediately that

$$\lim_{\lambda \to \infty} D' = I;$$

thus in this case we have a canonical normalisation condition.

Keeping in mind this example we go over to the more complicated case of WKI SP which reads (Wadati *et al* 1979)

$$\psi_x = \begin{pmatrix} i\lambda & \lambda q(x) \\ -\lambda q^*(x) & -i\lambda \end{pmatrix} \psi.$$

In this case, instead of formula (7), valid for the Zakharov–Shabat SP, we have more complicated formula

$$\psi^{(+)}(x, \lambda; q, q^*) \sim \begin{pmatrix} 1 & i(\rho - 1)/q^* \\ i(\rho - 1)/q & 1 \end{pmatrix} \begin{pmatrix} e^{\mu_-} & 0 \\ 0 & e^{-\mu^*} \end{pmatrix} \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix} [I + O(\lambda^{-1})] \times \begin{pmatrix} e^{-i\lambda\varepsilon_-} & 0 \\ 0 & e^{-i\lambda\varepsilon_+} \end{pmatrix} \tag{8}$$

where

$$\rho = (1 + |q|^2)^{1/2}, \qquad \mu_\pm = \pm \int_x^{\pm\infty} \left[\frac{1}{2} \frac{q_{x'}}{q} \left(\frac{\rho - 1}{\rho} \right) - \tfrac{1}{2} (\ln \rho)_{x'} \right] dx',$$

$$\varepsilon_\pm = \pm \int_x^{\pm\infty} (1 - \rho)\, dx'.$$

The formula (8), in view of the previous discussion of the Zakharov–Shabat case, strongly suggests the following two-sided gauge transformation

$$\psi = \begin{pmatrix} 1 & i(\rho - 1)/q^* \\ i(\rho - 1)/q & 1 \end{pmatrix} \begin{pmatrix} e^{\mu_-} & 0 \\ 0 & e^{-\mu_+^*} \end{pmatrix} \hat{\psi} \begin{pmatrix} e^{-i\lambda\varepsilon_-} & 0 \\ 0 & e^{-i\lambda\varepsilon_+} \end{pmatrix}.$$

The linear problem for $\hat{\psi}$ can be written in the form

$$\hat{\psi}_x = \rho \begin{pmatrix} i\lambda & Q \\ -Q^* & -i\lambda \end{pmatrix} \hat{\psi} + i\lambda(1 - \rho)\hat{\psi}\sigma_3$$

with

$$Q = \frac{i}{2} \left(\frac{\rho + 1}{\rho} \right) \left(\frac{1 - \rho}{q^*} \right)_x \exp(\mu_+^* + \mu_-).$$

Moreover, since now

$$\hat{\psi} \sim \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix}$$

as λ goes to infinity, one has a canonical normalisation condition, i.e. $D_1 = I$.

The detailed analysis of the construction of the Darboux matrix for the WKI case will be discussed in more detail in a forthcoming paper (Levi *et al* 1984).

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